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Holomorphic Monsters

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Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$ be an open set with simply connected components. For a function φ which is holomorphic on $\mathcal{O}$, three new types of cluster sets are introduced; the classical cluster sets with values in $\hat{\mathbb{C}}$ are replaced by certain cluster sets with values in function spaces. We deal with the construction of holomorphic functions φ on $\mathcal{O}$ such that φ and all its derivatives and antiderivatives have maximal cluster sets of any of these types at any boundary point of $\mathcal{O}$ ("holomorphic monsters"). It is also shown that the family of all holomorphic monsters is a dense subset in the space of all holomorphic functions on $\mathcal{O}$ established with a natural metric. © 1988 Academic Press, Inc.

1. INTRODUCTION

1.1. *Motivation: Classical Cluster Sets*

Let $G \subset \mathbb{C}$ be a domain, suppose that ζ is a boundary point of G and let the function φ be holomorphic on G. The classical cluster set $S(\varphi, G, \zeta)$ is defined as the set of all numbers $w \in \hat{\mathbb{C}}$ for which a sequence of points $\{z_n\}_{n \in \mathbb{N}}$ exists, having the following properties

$$z_n \in G \text{ for all } n \in \mathbb{N}; \qquad z_n \to \zeta, \qquad f(z_n) \to w \qquad \text{for} \quad n \to \infty.$$

It is clear that $S(\varphi, G, \zeta)$ is always a nonempty subset of $\hat{\mathbb{C}}$ and if, e.g., ζ is an essential singularity of φ then the corresponding cluster set is maximal; i.e., we have $S(\varphi, G, \zeta) = \hat{\mathbb{C}}$.

There is an extensive literature on the field of cluster sets. A summary of the classical results can be found in [22]. In this paper we deal with several new types of modified cluster sets and the construction of holomorphic functions having maximal cluster sets. Our results give good insight into the boundary behaviour as well as into the approximation properties of holomorphic functions.

One might ask the following question. What is the behaviour of the holomorphic function φ in the domain G near the boundary point ζ if the

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sequence $\{z_n\}$ occurring above is made slightly dependent on a variable z? To be precise: we shall investigate the behaviour of the sequence of functions $\varphi(a_n z + b_n)$, where z belongs to a certain subset S of $\mathbb{C}$ (to be specified later) and $a_n z + b_n \in G$ for all $z \in S$ and all $n \in \mathbb{N}$; $a_n \to 0$, $b_n \to \zeta$ for $n \to \infty$.

The same question arises if we consider, instead of a domain G, a more general open set $\mathcal{O} \subset \mathbb{C}$ and a holomorphic function φ on $\mathcal{O}$ (which means that the restriction of φ to any component of $\mathcal{O}$ is holomorphic on this component).

It is natural to ask if there are functions φ on open sets $\mathcal{O} \subset \mathbb{C}$ and functions f on sets $S \subset \mathbb{C}$ such that f can be approximated by the values of φ near a boundary point $\zeta \in \partial\mathcal{O}$ in the sense that

$$\lim_{n \to \infty} \varphi(a_n z + b_n) = f(z)$$

holds pointwise or uniformly or almost everywhere on S.

1.2. *Notations*

Let $S \subset \mathbb{C}$ be any set. By $H(S)$ we denote the family of all functions which are holomorphic on S, whereas $A(S)$ consists of all functions which are continuous on S and holomorphic in the interior $\mathring{S}$ of S. If S is a Lebesgue-measurable set in $\mathbb{C}$ then we denote by $\mu(S)$ its 2-dimensional (Lebesgue-) measure and by $M(S)$ the family of all (Lebesgue-) measurable functions on S.

Let $\{f_n\}$ be a sequence of functions, defined on a set S. If it converges uniformly on S to a function f, we use the notation $f_n(z) \rightrightarrows_B f(z)$. The sequence will be called compactly convergent on S to f (notation $f_n(z) \Rrightarrow_S f(z)$) if it converges to f uniformly on every compact subset of S. Finally we write $f_n(z) \to_S^{\text{a.e.}} f(z)$ if it converges almost everywhere to f on S.

Suppose that the function f is holomorphic on an open set $\mathcal{O} \subset \mathbb{C}$ having simply connected components. If $j \in \mathbb{N}_0$ we denote as usual by $f^{(j)}$ the derivative of f with order j. If $-j \in \mathbb{N}$ we use the abbreviation $f^{(j)}$ for an (arbitrary but fixed) antiderivative of f with order $-j$; i.e., we have for all $z \in \mathcal{O}$

$$\frac{d^{-j}}{dz^{-j}} f^{(j)}(z) = f(z).$$

Any other antiderivatives of f with order $-j$ differs from $f^{(j)}$ on each component of $\mathcal{O}$ by a certain polynomial of degree less than $-j$.

1.3. *Modified Cluster Sets*

Let us now define three types of cluster sets. The classical cluster sets with values in $\hat{\mathbb{C}}$ are replaced by certain cluster sets with values in function spaces. We always assume that $\mathcal{O} \subset \mathbb{C}$ is an open set, that the function φ is holomorphic on $\mathcal{O}$ and that ζ is a boundary point of $\mathcal{O}$.

Cluster sets of type 1. Suppose $B \subset \mathbb{C}$ is a compact set. By $S(\varphi, \mathcal{O}, \zeta, B)$ we denote the set of all functions $f\colon B \to \mathbb{C}$ for which there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b\}_{n\in\mathbb{N}}$ such that $a_n z + b_n \in \mathcal{O}$ for all $z \in B$ and all $n \in \mathbb{N}$; $a_n \to 0$, $b_n \to \zeta$ for $n \to \infty$ and

$$\varphi(a_n z + b_n) \underset{B}{\Longrightarrow} f(z) \qquad \text{for} \quad n \to \infty.$$

It is clear that $S(\varphi, \mathcal{O}, \zeta, B)$ is a subset of $A(B)$.

Cluster sets of type 2. Suppose $U \subset \mathbb{C}$ is an open set. By $R(\varphi, \mathcal{O}, \zeta, U)$ we denote the set of all functions $f\colon U \to \mathbb{C}$ for which there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that $a_n z + b_n \in \mathcal{O}$ for all $z \in U$ and all $n \in \mathbb{N}$; $a_n \to 0$, $b_n \to \zeta$ for $n \to \infty$ and

$$\varphi(a_n z + b_n) \underset{U}{\Rrightarrow} f(z) \qquad \text{for} \quad n \to \infty.$$

It is clear that $R(\varphi, \mathcal{O}, \zeta, U)$ is a subset of $H(U)$.

Cluster sets of type 3. Suppose $S \subset \mathbb{C}$ is a measurable set. By $T(\varphi, \mathcal{O}, \zeta, S)$ we denote the set of all functions $f\colon S \to \hat{\mathbb{C}}$ for which there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that $a_n z + b_n \in \mathcal{O}$ for all $z \in S$ and all $n \in \mathbb{N}$; $a_n \to 0$, $b_n \to \zeta$ for $n \to \infty$ and

$$\varphi(a_n z + b_n) \xrightarrow[S]{\text{a.e.}} f(z) \qquad \text{for} \quad n \to \infty.$$

It is clear that $T(\varphi, \mathcal{O}, \zeta, S)$ is a subset of $M(S)$.

Assume that $\mathcal{O} = G$, $G \neq \mathbb{C}$, is a simply connected domain. Then in [18] a "universal" function $\varphi \in H(G)$ has been constructed such that $S(\varphi, G, \zeta, B) = A(B)$ holds for all $\zeta \in \partial G$ and all compact sets B with connected complement; i.e., the cluster sets of type 1 for φ in this domain G are maximal for all $\zeta \in \partial G$ and all those B. Kanatnikov [12] extended this result in a certain direction for meromorphic functions and several modified cluster sets.

The problems of the existence of "universal" functions and their correspondence with the "universal approximation" of functions are classical; cf. [19] where further references are given. In recent papers the problem of universal approximation has been investigated by Voronin [34] who proved that Riemann's ζ-function has a remarkable approximation property; extensions and variants of this result are due to Reich [25, 26] and Gavrilov and Kanatnikov [10]. Universal approximation via overconvergence of power series has been studied by Chui and Parnes [3], Luh [14, 15], Tomm and Trautner [33]. In some works universal approximation related with problems in summability theory has been investigated by Luh [16], Luh and Trautner [17], Faulstich and Luh [7], Faulstich, Luh and Tomm [6].

In this paper we deal with the construction of function φ, holomorphic on open sets $\mathcal{O} \subset \mathbb{C}$ with simply connected components, such that φ and all its derivatives and all its antiderivatives have maximal cluster sets of types 1, 2, and 3 at any boundary point of $\mathcal{O}$ (Thoerem 4). The basic tools for the construction of such "holomorphic monsters" are the approximation theorems of Runge and Mergelyan (and it seems to us that the use of these results is a powerful method for the construction of holomorphic functions with prescribed boundary behaviour, cf. Gaier [8; Chap. IV, Section 5] and Schneider [28]). We shall show (Theorem 5) that the family of all holomorphic monsters is a dense subset in the space $H(\mathcal{O})$, established with a natural metric.

2. Functions with Maximal Cluster Sets

2.1. *The Main Results*

We first consider a simply connected domain $G \neq \mathbb{C}$ and prove the existence of a universal holomorphic function on G which has maximal cluster sets of type 1.

THEOREM 1. *Let $G \subset \mathbb{C}$, $G \neq \mathbb{C}$ be a simply connected domain. Then there exists a function $\varphi \in H(G)$ with the following properties. For all $\zeta \in \partial G$, for all compact sets B with connected complement and for all derivatives $\varphi^{(j)}$ $(j \in \mathbb{N}_0)$ and all antiderivatives $\varphi^{(j)}$ $(-j \in \mathbb{N})$ we have*

$$S(\varphi^{(j)}, G, \zeta, B) = A(B).$$

Proof. 1. Without loss of generality we may assume that $\mathbb{D} := \{z : |z| < 1\} \subset G$. Let $\{\zeta^{(k)}\}_{k \in \mathbb{N}}$ be a sequence of boundary-points of G which is dense in ∂G.

We choose a sequence of Jordan domains $\{G_n\}_{n \in \mathbb{N}}$ having the properties that for all $n \in \mathbb{N}$ the boundary ∂G_n is rectifiable (its length is denoted by L_n) that $\bar{G}_n \subset G_{n+1} \subset G$ and that for any compact set $K \subset G$ there exists an $n_0 = n_0(K) \in \mathbb{N}$ such that $K \subset G_n$ for all $n > n_0$.

For each $j \in \mathbb{Z}$ and each $k \in \mathbb{N}$ we can find a sequence $\{z_n^{(k,j)}\}_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} z_n^{(k,j)} = \zeta^{(k)}$ for all $j \in \mathbb{Z}$ and all $k \in \mathbb{N}$ such that for fixed n and $k = 1, ..., n$; $j = 0, \pm 1, \pm 2, ..., \pm n$ the points $z_n^{(k,j)}$ belong to $G_{n+1} \setminus \bar{G}_n$ and are pairwise disjoint.

Next we choose for any $n \in \mathbb{N}$ a real number $r_n \in (0, 1/n)$ so small that for $k = 1, ..., n$ and $j = 0, \pm 1, \pm 2, ..., \pm n$ the closed disks

$$D_n^{(k,j)} := \{z : |z - z_n^{(k,j)}| \leqslant r_n\}$$

are pairwise disjoint and that

$$D_n := \bigcup_{\substack{k=1, ..., n \\ j=0, \pm 1, ..., \pm n}} D_n^{(k,j)} \subset G_{n+1} \backslash \bar{G}_n.$$

Obviously D_n is a compact set and its complement is connected. We define

$$d_0 := 1, \qquad d_n := \min\{\operatorname{dist}(D_n, \partial G_{n+1}), \operatorname{dist}(\bar{G}_n, \partial G_{n+1}), 1\} \tag{2.1}$$

and

$$\varepsilon_n := \min\left\{\frac{2\pi(d_{n-1})^{2n}}{(2n-1)!\, L_n n^2}, \frac{(r_n)^{2n}}{(2n)!\, 2^{2n+1} n}\right\}. \tag{2.2}$$

2. Let $\{Q_n(z)\}_{n \in \mathbb{N}}$ be the sequence of polynomials whose coefficients have rational real- and imaginary-parts. For $k = 1, ..., n$ and $j = 0, \pm 1, \pm 2, ..., \pm n$ denote by $F_n^{(k,j)}(w)$ a polynomial with the property

$$\frac{d^{j+n}}{dw^{j+n}} F_n^{(k,j)}(w) = Q_n\left(\frac{2n}{(n+1)\, r_n} (w - z_n^{(k,j)})\right). \tag{2.3}$$

Next we construct a sequence of polynomials $\Pi_n(w)$. Let $\Pi_0(w) \equiv 0$ and assume that for an $n \in \mathbb{N}$ the polynomials $\Pi_0(w), ..., \Pi_{n-1}(w)$ have already been determined and suppose that $\tilde{\Pi}_{n-1}(w)$ is any polynomial satisfying

$$\tilde{\Pi}'_{n-1}(w) = \Pi_{n-1}(w).$$

By Runge's approximation theorem we can find a polynomial $\Pi_n(w)$ with the following properties: for $k = 1, ..., n$ and $j = 0, \pm 1, ..., \pm n$ we have

$$\max_{D_n^{(k,j)}} |\Pi_n(w) - F_n^{(k,j)}(w)| < \varepsilon_n, \tag{2.4}$$

$$\max_{\bar{G}_n} |\Pi_n(w) - \tilde{\Pi}_{n-1}(w)| < \varepsilon_n. \tag{2.5}$$

Now we define

$$P_n(w) := P_n^{(0)}(w) := \frac{d^n}{dw^n} \Pi_n(w)$$

and

$$\varphi(w) := \sum_{n=1}^{\infty} \{P_n(w) - P_{n-1}(w)\}. \tag{2.6}$$

If $w \in G_n$ is fixed we obtain

$$P_n(w) - P_{n-1}(w) = \frac{n!}{2\pi i} \int_{\partial G_n} \frac{\Pi_n(z) - \tilde{\Pi}_{n-1}(z)}{(z-w)^{n+1}} \, dz.$$

We therefore get by (2.1), (2.2), and (2.5) for all $n \geqslant 2$

$$\max_{\overline{G_{n-1}}} |P_n(w) - P_{n-1}(w)| < \frac{1}{n^2}.$$

This estimation implies that the series (2.6) converges compactly on G and hence we have $\varphi \in H(G)$.

3. Now let j be a fixed integer. We consider the functions

$$P_0^{(j)}(w) \equiv 0,$$

$$P_n^{(j)}(w) := \frac{d^{n+j}}{dw^{n+j}} \Pi_n(w) \qquad (n \in \mathbb{N}),$$

$$\varphi_0^{(j)}(w) := \sum_{\nu=1}^{\infty} \{P_\nu^{(j)}(w) - P_{\nu-1}^{(j)}(w)\}.$$

(It follows easily that this series is compactly convergent on G.) If $j \geqslant 0$ then $\varphi_0^{(j)}(w)$ is the jth derivative of $\varphi(w)$ and if $j < 0$ then $\varphi_0^{(j)}(w)$ denotes an antiderivative of $\varphi(w)$ with order $-j$. Any other antiderivatives of order $-j$ differ from $\varphi_0^{(j)}(w)$ by a polynomial of degree less than $-j$.

For fixed $k \in \mathbb{N}$ and $n \in \mathbb{N}$ we compute

$$\begin{aligned}
&\varphi_0^{(j)}(w) - Q_n\left(\frac{2n}{(n+1)\, r_n} \cdot (w - z_n^{(k,j)})\right) \\
&\quad = \sum_{\nu=n}^{\infty} \{P_{\nu+1}^{(j)}(w) - P_\nu^{(j)}(w)\} \\
&\qquad + \left\{P_n^{(j)}(w) - Q_n\left(\frac{2n}{(n+1)\, r_n} \cdot (w - z_n^{(k,j)})\right)\right\},
\end{aligned}$$

and for $\nu \geqslant n > |j|$ we obtain by (2.1), (2.2), (2.5) the following estimation

$$\begin{aligned}
&\max_{|w - z_n^{(k,j)}| \leqslant r_n/2} |P_{\nu+1}^{(j)}(w) - P_\nu^{(j)}(w)| \\
&\quad \leqslant \max_{D_n} \left| \frac{(\nu+1+j)!}{2\pi i} \int_{\partial G_{\nu+1}} \frac{\Pi_{\nu+1}(z) - \tilde{\Pi}_\nu(z)}{(z-w)^{\nu+2+j}} \, dz \right| < \frac{1}{(\nu+1)^2},
\end{aligned}$$

and for $n > |j|$ we get by (2.2), (2.3), and (2.4)

$$\max_{|w - z_n^{(k,j)}| \leqslant r_n/2} \left| P_n^{(j)}(w) - Q_n\left(\frac{2n}{(n+1)\, r_n}(w - z_n^{(k,j)})\right)\right|$$

$$= \max_{|w - z_n^{(k,j)}| \leqslant r_n/2} \left| \frac{(n+j)!}{2\pi i} \int_{\partial D_n^{(k,j)}} \frac{\Pi_n(z) - F_n^{(k,j)}(z)}{(z-w)^{n+j+1}}\, dz \right| < \frac{1}{n}.$$

Summing up we obtain for $n > |j|$

$$\max_{|w - z_n^{(k,j)}| \leqslant r_n/2} \left| \varphi_0^{(j)}(w) - Q_n\left(\frac{2n}{(n+1)\, r_n}(w - z_n^{(k,j)})\right)\right| < \frac{2}{n}$$

or, equivalently,

$$\max_{|z| \leqslant n/(n+1)} \left| \varphi_0^{(j)}\left(\frac{r_n(n+1)}{2n} z + z_n^{(k,j)}\right) - Q_n(z)\right| < \frac{2}{n}. \tag{2.7}$$

4. Now let be given any $\zeta \in \partial G$, any compact set B with connected complement, any function $f \in A(B)$ and any $j \in \mathbb{Z}$. We choose a real number $R \geqslant 1$ such that

$$B_R := \left\{ z \colon z = \frac{z^*}{R},\ z^* \in B \right\} \subset \mathbb{D}.$$

The function f_R with $f_R(z) := f(Rz)$ belongs to $A(B_R)$ and hence by the theorem of Mergelyan there exists a sequence $\{n_k\}_{k \in \mathbb{N}}$ of natural numbers with $n_k \to \infty$ for $k \to \infty$ and

$$\max_{B_R} |f_R(z) - Q_{n_k}(z)| < 1/k \qquad (k = 1, 2, \ldots). \tag{2.8}$$

For sufficiently large k we have $B_R \subset \{z \colon |z| \leqslant n_k/(n_k+1)\}$ and $n_k > |j|$. The set of points $z_{n_k}^{(\mu,j)}$ has the limit point ζ, so we can find sequences $\{\mu_s\}_{s \in \mathbb{N}}$ and $\{k_s\}_{s \in \mathbb{N}}$ with $k_s \geqslant s$ and

$$\lim_{s \to \infty} z_{n_{k_s}}^{(\mu_s, j)} = \zeta.$$

If we choose

$$a_s := \frac{n_{k_s+1}}{n_{k_s}} \cdot \frac{r_{n_{k_s}}}{2} \cdot \frac{1}{R}, \qquad b_s := z_{n_{k_s}}^{(\mu_s, j)}$$

we get $Ra_s z+b_s \in G$ for $z \in B_R$, $a_s \to 0$ (independent of j) and $b_s \to \zeta$ for $s \to \infty$ and by (2.7) and (2.8) we obtain

$$\max_{B_R} |\varphi_0^{(j)}(Ra_s z+b_s)-f_R(z)| \to 0 \qquad (s \to \infty)$$

or, equivalently,

$$\varphi_0^{(j)}(a_s z+b_s) \underset{B}{\Longrightarrow} f(z) \qquad (s \to \infty).$$

5. If $j \geqslant 0$ then we have $\varphi_0^{(j)}(z)=\varphi^{(j)}(z)$ and our assertion follows. In case of $j<0$ let any antiderivative $\varphi^{(j)}(z)$ to $\varphi(z)$ of order $-j$ be given. Then there exists a polynomial $P(z)$ of degree less than $-j$ such that

$$\varphi^{(j)}(z)=\varphi_0^{(j)}(z)+P(z) \qquad \text{for all} \quad z \in G.$$

Let any compact set B with connected complement, any functon $f \in A(B)$, and any boundary point ζ of G be given.

If $\zeta \neq \infty$ we can choose sequences $\{a_s\}_{s \in \mathbb{N}}$ and $\{b_s\}_{s \in \mathbb{N}}$ such that $a_s z+b_s \in G$ for all $z \in B$ and all $n \in \mathbb{N}$; $a_s \to 0$, $b_s \to \zeta$ for $s \to \infty$ and

$$\varphi_0^{(j)}(a_s z+b_s) \underset{B}{\Longrightarrow} f(z)-P(\zeta) \qquad (s \to \infty).$$

We therefore obtain

$$\varphi^{(j)}(a_s z+b_s) \underset{B}{\Longrightarrow} f(z) \qquad (s \to \infty).$$

If $\zeta=\infty$ then we can find a sequence $\{\zeta_m\}_{m \in \mathbb{N}}$ with $\zeta_m \in \mathbb{C}$, $\zeta_m \in \partial G$, and $\zeta_m \to \infty$ for $m \to \infty$. For any $m \in \mathbb{N}$ there are sequences $\{a_s^{(m)}\}_{s \in \mathbb{N}}$ and $\{b_s^{(m)}\}_{s \in \mathbb{N}}$ such that $a_s^{(m)} z+b_s^{(m)} \in G$ for all $s \in \mathbb{N}$ and all $z \in B$; $a_s^{(m)} \to 0$, $b_s^{(m)} \to \zeta_m$ for $s \to \infty$ and

$$\varphi_0^{(j)}(a_s^{(m)} z+b_s^{(m)}) \underset{B}{\Longrightarrow} f(z)-P(\zeta_m) \qquad (s \to \infty).$$

We choose an integer $s_m>m$ so that the following conditions hold

$$|a_{s_m}^{(m)}|<\frac{1}{m}, \qquad |b_{s_m}^{(m)}-\zeta_m|<\frac{1}{m};$$

$$\max_B |P(a_{s_m}^{(m)} z+b_{s_m}^{(m)})-P(\zeta_m)|<\frac{1}{m};$$

$$\max_B |\varphi_0^{(j)}(a_{s_m}^{(m)} z+b_{s_m}^{(m)})-f(z)+P(\zeta_m)|<\frac{1}{m}.$$

It follows that $\alpha_m := a_{s_m}^{(m)} \to 0$, $\beta_m := b_{s_m}^{(m)} \to \zeta = \infty$, and by a simple estimation we get

$$\varphi^{(j)}(\alpha_m z + \beta_m) \underset{B}{\Longrightarrow} f(z) \qquad (m \to \infty).$$

Our theorem is proved.

We next consider more generally an open set $\mathcal{O} \neq \mathbb{C}$ with simply connected components and prove by a simple application of Theorem 1 the existence of a universal holomorphic function on $\mathcal{O}$ which has maximal cluster sets of type 1.

THEOREM 2. *Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$ be an open set with simply connected components. Then there exists a function $\varphi \in H(\mathcal{O})$ with the following properties. For all $\zeta \in \partial\mathcal{O}$, for all compact sets B with connected complement and for all derivatives $\varphi^{(j)}$ $(j \in \mathbb{N}_0)$ and all antiderivatives $\varphi^{(j)}$ $(-j \in \mathbb{N})$ we have*

$$S(\varphi^{(j)}, \mathcal{O}, \zeta, B) = A(B).$$

Proof. There exists a finite or countable set I such that $\mathcal{O} = \bigcup_{i \in I} G_i$, where the G_i are pairwise disjoint simply connected domains in $\mathbb{C}$. For any $i \in I$ we choose a function $\varphi_i \in H(G_i)$ with the properties as in Theorem 1. We define the functon $\varphi \in H(\mathcal{O})$ by $\varphi(z) := \varphi_i(z)$ if $z \in G_i$ and we shall show that φ has the required properties. To this end let there be given any $\zeta \in \partial\mathcal{O}$, any compact set B with connected complement, any function $f \in A(B)$, and any derivative $\varphi^{(j)}$ $(j \in \mathbb{N}_0)$ or any antiderivative $\varphi^{(j)}$ $(-j \in \mathbb{N})$. We have

$$\varphi^{(j)}(z) = \varphi_i^{(j)}(z) \qquad \text{for} \quad z \in G_i \qquad (i \in I, j \in \mathbb{Z}).$$

If $j < 0$ then $\varphi_i^{(j)}$ is a suitable antiderivative with order $-j$ of φ_i on G_i.

1. We first suppose that there exists an $i_0 \in I$ such that $\zeta \in \partial G_{i_0}$. Then by Theorem 1 there are sequences $\{a_n\}$ and $\{b_n\}$ (depending on ζ, B, f, $\varphi_{i_0}^{(j)}$) such that $a_n z + b_n \in G_{i_0}$ for all $z \in B$ and all $n \in \mathbb{N}$ and the property

$$\varphi^{(j)}(a_n z + b_n) = \varphi_{i_0}^{(j)}(a_n z + b_n) \underset{B}{\Longrightarrow} f(z) \qquad (n \to \infty).$$

2. If $\zeta \notin \partial G_i$ for all $i \in I$ then there must be a sequence $\{i_k\}_{k \in \mathbb{N}}$ and boundary points $\zeta_k \in \partial G_{i_k}$ with $\zeta_k \in \mathbb{C}$ and $\zeta_k \to \zeta$ for $k \to \infty$. By Theorem 1 we can choose for any $k \in \mathbb{N}$ sequences $\{a_n^{(k)}\}_{n \in \mathbb{N}}$ and $\{b_n^{(k)}\}_{n \in \mathbb{N}}$ (depending on ζ_k, B, f, $\varphi_{i_k}^{(j)}$) such that $a_n^{(k)} z + b_n^{(k)} \in G_{i_k}$ for all $z \in B$ and all $n \in \mathbb{N}$, $a_n^{(k)} \to 0$, $b_n^{(k)} \to \zeta_k$ for $n \to \infty$, and

$$\varphi_{i_k}^{(j)}(a_n^{(k)} z + b_n^{(k)}) \underset{B}{\Longrightarrow} f(z) \qquad (n \to \infty).$$

For fixed $k \in \mathbb{N}$ we can choose an integer $n_k > k$ such that the following conditions hold

$$|a_{n_k}^{(k)}| < \frac{1}{k}, \qquad |b_{n_k}^{(k)} - \zeta_k| < \frac{1}{k},$$

$$\max_B |\varphi_{i_k}^{(j)}(a_{n_k}^{(k)} z + b_{n_k}^{(k)}) - f(z)| < \frac{1}{k}.$$

We therefore obtain $a_{n_k}^{(k)} \to 0$, $b_{n_k}^{(k)} \to \zeta$ for $k \to \infty$ and

$$\varphi^{(j)}(a_{n_k}^{(k)} z + b_{n_k}^{(k)}) \underset{B}{\Rightarrow} f(z) \qquad (k \to \infty)$$

which proves our theorem.

2.2. *Cluster Sets of Types* 2 *and* 3

We first prove the following result.

THEOREM 3. *Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$ be an open set with simply connected components and assume $\zeta \in \partial\mathcal{O}$ is fixed. Suppose $\varphi \in H(\mathcal{O})$ satisfies $S(\varphi, \mathcal{O}, \zeta, B) = A(B)$ for all compact sets B with connected complement. Then the function φ has, moreover, the following properties*:

(1) *For all open sets $U = \mathbb{C}$ with simply connected components we have $R(\varphi, \mathcal{O}, \zeta, U) = H(U)$.*

(2) *For all measurable sets $S \subset \mathbb{C}$ we have $T(\varphi, \mathcal{O}, \zeta, S) = M(S)$.*

Proof. 1. Let any open set $U \subset \mathbb{C}$ with simply connected components and any function $f \in H(U)$ be given. For an $\varepsilon > 0$ we define $N_\varepsilon(\zeta) := \{z: |z - \zeta| < \varepsilon\}$ if $\zeta \in \mathbb{C}$ and $N_\varepsilon(\zeta) := \{z: |z| > 1/\varepsilon\}$ if $\zeta = \infty$. There exists a finite or countable set J such that $U = \bigcup_{j \in J} U_j$, where the U_j $(j \in J)$ are pairwise disjoint simply connected domains in $\mathbb{C}$. For any U_j we can choose an exhausting sequence of Jordan domains $\{U_{jn}\}_{n \in \mathbb{N}}$ such that $\overline{U_{jn}} \subset U_{j,n+1} \subset U_j$ for all $n \in \mathbb{N}$ and that any compact subset of U_j is contained in U_{jn} for all sufficiently large n. The sets

$$K_n := \bigcup_{\substack{j \in J \\ j \leq n}} \overline{U_{jn}} \qquad (n \in \mathbb{N})$$

are compact and have a connected complement. We have $f \in A(K_n)$. Given $n \in \mathbb{N}$, then by the assumptions made on the function φ we can choose sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ with $|a_n|<1/n$, $b_n\in U_{1/n}(\zeta)$ for all $n\in\mathbb{N}$ such that $a_nz+b_n\in\mathcal{O}$ for all $z\in K_n$, $n\in\mathbb{N}$, and

$$\max_{K_n}|\varphi(a_nz+b_n)-f(z)|<\frac{1}{n}. \tag{2.9}$$

It follows by this construction that $\varphi(a_nz+b_n)\Rightarrow_U f(z)$ and assertion (1) is proved.

2. Now let any measurable $S\subset\mathbb{C}$ and any function $g\in M(S)$ be given. The function

$$f(z):=\begin{cases} g(z) & \text{if } z\in S\\ 0 & \text{if } z\notin S\end{cases}$$

is measurable on $\mathbb{C}$ and it suffices to approximate f almost everywhere on $\mathbb{C}$.

Let there be given an $n\in\mathbb{N}$ and define $\mathbb{D}_n:=\{z\colon|z|<n\}$. Suppose that the function f_n is defined on $\mathbb{D}_n$ by $f_n(z):=f(z)$ if $f(z)\neq\infty$ and by $f_n(z):=n$ if $f(z)=\infty$. According to Lusin's theorem there exists a measurable set $L_n\subset\bar{\mathbb{D}}_n$ with $\mu(\bar{\mathbb{D}}_n\backslash L_n)\leqslant 1/2^{n+1}$ and a continuous function g_n on L_n with $g_n(z)=f_n(z)$ for all $z\in L_n$. The set $L_n^*:=\{z\colon z\in L_n,\ \mathrm{Re}(z)\notin\mathbb{Q},\ \mathrm{Im}(z)\notin\mathbb{Q}\}$ has empty interior and satisfies $\mu(L_n^*)=\mu(L_n)$.

We may choose a compact set $M_n\subset L_n^*$ such that

$$\mu(L_n^*\backslash M_n)<\frac{1}{2^{n+2}}. \tag{2.10}$$

There exists a finite or countable set J_n of natural numbers such that $M_n^c=\bigcup_{j\in J_n}G_n^{(j)}$, where the $G_n^{(j)}$ denote the components of M_n^c. For any $j\in J_n$ choose a point $z_n^{(j)}\in G_n^{(j)}$ and suppose

$$z_n^{(j)}:=r_n^{(j)}\cdot e^{i\theta_n^{(j)}}\qquad\text{where}\quad r_n^{(j)}>0,\ \theta_n^{(j)}\in\mathbb{R}.$$

Now consider the sets

$$H_n^{(j)}:=\left\{z=\rho e^{i\theta}\colon\frac{1}{2}r_n^{(j)}<\rho<2n;\ |\theta-\theta_n^{(j)}|<\frac{1}{n^2\cdot 2^{n+j+4}}\right\},$$

$$K_n:=\left\{\bar{\mathbb{D}}_n^c\cup\bigcup_{j\in J_n}(G_n^{(j)}\cup H_n^{(j)})\right\}^c;$$

K_n is a compact set with connected complement and, since $K_n\subset M_n\subset L_n^*$, the interior of K_n is empty. We have

$$\mu(M_n\backslash K_n)\leqslant\mu\left(\bigcup_{j\in J_n}H_n^{(j)}\right)\leqslant\frac{1}{2^{n+2}}$$

and hence we obtain with (2.10)

$$\begin{aligned}\mu(\bar{\mathbb{D}}_n\backslash K_n) &\leqslant \mu(\bar{\mathbb{D}}_n\backslash L_n)+\mu(L_n^*\backslash M_n)+\mu(M_n\backslash K_n)\\ &\leqslant \frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}+\frac{1}{2^{n+2}}=\frac{1}{2^n}.\end{aligned} \tag{2.11}$$

By the assumptions made on the function φ we can choose sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ with $|\alpha_n|<1/n$, $\beta_n\in N_{1/n}(\zeta)$ for all $n\in\mathbb{N}$ such that $\alpha_n z+\beta_n\in\mathcal{O}$ for all $z\in K_n$, $n\in\mathbb{N}$, and

$$\max_{K_n} |\varphi(\alpha_n z+\beta_n)-f_n(z)|<\frac{1}{n}. \tag{2.12}$$

Consider the set $K:=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} K_k$. Using (2.11) we compute

$$\mu(\bar{\mathbb{D}}_n\backslash K)\leqslant\mu\left(\bigcup_{k=n}^{\infty}(\bar{\mathbb{D}}_k\backslash K_k)\right)\leqslant\frac{1}{2^{n-1}}$$

and therefore we get

$$\mu(\mathbb{C}\backslash K)=\lim_{n\to\infty}\mu(\bar{\mathbb{D}}_n\backslash K)=0.$$

Hence our assertion will follow if we can prove that

$$\varphi(\alpha_n z+\beta_n)\to f(z) \qquad \text{for all} \quad z\in K \quad \text{and} \quad n\to\infty.$$

Suppose $z\in K$ is given, then we can choose an n_0 such that $z\in K_n$ for all $n\geqslant n_0$. If $f(z)\neq\infty$ we obtain for all $n\geqslant n_0$ by (2.12)

$$|\varphi(\alpha_n z+\beta_n)-f(z)|\leqslant\max_{K_n}|\varphi(\alpha_n z+\beta_n)-f_n(z)|<\frac{1}{n}.$$

If $f(z)=\infty$ we obtain for all $n\geqslant n_0$ by (2.12)

$$\begin{aligned}|\varphi(\alpha_n z+\beta_n)| &\geqslant n-|\varphi(\alpha_n z+\beta_n)-f_n(z)|\\ &\geqslant n-\max_{K_n}|\varphi(\alpha_n z+\beta_n)-f_n(z)|\geqslant n-\frac{1}{n}.\end{aligned}$$

This proves assertion (2).

Combining Therem 2 and Theorem 3 we obtain as a simple corollary the following result.

THEOREM 4. *Let $\mathcal{O}\subset\mathbb{C}$, $\mathcal{O}\neq\mathbb{C}$ be an open set with simply connected components. Then there exists a function $\varphi\in H(\mathcal{O})$ with the following proper-*

ties. For all $\zeta \in \partial\mathcal{O}$ *all derivatives* $\varphi^{(j)}$ $(j \in \mathbb{N}_0)$ *and all antiderivatives* $\varphi^{(j)}$ $(-j \in \mathbb{N})$ *satisfy*:

(1) $S(\varphi^{(j)}, \mathcal{O}, \zeta, B) = A(B)$ *for all compact sets B with connected complement*;

(2) $R(\varphi^{(j)}, \mathcal{O}, \zeta, U) = H(U)$ *for all open sets U with simply connected components*;

(3) $T(\varphi^{(j)}, \mathcal{O}, \zeta, S) = M(S)$ *for all measurable sets S.*

Clearly the cluster sets which occur in (1)–(3) are maximal and the assumptions made on B, U, S, respectively, cannot be weakened.

The function φ, constructed in Theorem 2 has—roughly speaking—the property that all functions which belong to certain natural spaces can be approximated by the values of φ or any derivative or any antiderivative near any boundary point of $\mathcal{O}$. We therefore give the following definition.

DEFINITION. Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$ be an open set with simply connected components. Any function which has the same properties as the function φ in Theorem 2 is called a holomorphic monster on $\mathcal{O}$. The set of all holomorphic monsters on $\mathcal{O}$ is denoted by $m(\mathcal{O})$.

3. $m(\mathcal{O})$ AS A SUBSET OF $H(\mathcal{O})$

We now deal with the question whether it might be considered as a "normal" feature that a holomorphic function is a holomorphic monster. In other words, we ask if $m(\mathcal{O})$ is a "big" subset of $H(\mathcal{O})$. To this end a certain natural metric in $H(\mathcal{O})$ is introduced such that $H(\mathcal{O})$ becomes a complete metric space. We will show that $m(\mathcal{O})$ is dense in $H(\mathcal{O})$.

3.1. *The Metric Space* $H(\mathcal{O})$

Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$ be as usual an open set with simply connected components and let $I = \{1, 2, ...,\}$ be a finite or countable set such that

$$\mathcal{O} = \bigcup_{i \in I} G_i,$$

where the G_i are pairwise disjoint simply connected components. For any $i \in I$ let $\{G_{in}\}_{n \in \mathbb{N}}$ be an exhausting sequence of Jordan domains with $\overline{G_{in}} \subset G_{i,n+1} \subset G_i$ for all n and the property that a compact subset of G_i is contained in G_{in} for all sufficiently large n. Then the sets

$$\mathcal{O}_n := \bigcup_{\substack{i \in I \\ i \leqslant n}} G_{in} \qquad (n \in \mathbb{N})$$

form an exhausting sequence for the open set $\mathcal{O}$. Let φ be a function which belongs to $H(\mathcal{O})$. For $n \in \mathbb{N}$ we define

$$d_n(\varphi) := \max_{\overline{\mathcal{O}_n}} |\varphi(z)|,$$

$$d(\varphi) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(\varphi)}{1 + d_n(\varphi)}$$

and if $f, g \in H(\mathcal{O})$ we define their distance by

$$d(f, g) := d(f - g). \tag{3.1}$$

It follows (compare [4; Chap. VII]) that the space $H(\mathcal{O})$ established with the metric (3.1) is a complete metric space. For a sequence of functions $f_k \in H(\mathcal{O})$ and $f \in H(\mathcal{O})$ we have $\lim_{k \to \infty} d(f_k, f) = 0$ if and only if $\{f_k(z)\}$ converges compactly to $f(z)$ on $\mathcal{O}$. This shows that d is a natural metric in $H(\mathcal{O})$, which is induced by the compact convergence.

We now compare the subset $m(\mathcal{O})$ with $H(\mathcal{O})$ and prove the following result.

THEOREM 5. *Let $\mathcal{O} \subset \mathbb{C}$, $\mathcal{O} \neq \mathbb{C}$ be an open set with simply connected components. Then the set $m(\mathcal{O})$ of holomorphic monsters is dense in $H(\mathcal{O})$.*

Proof. Let be given any function $f \in H(\mathcal{O})$ and any $\varepsilon > 0$. We have to show that there exists a fuction $\varphi \in m(\mathcal{O})$ with $d(f, \varphi) < \varepsilon$.

1. By Theorem 2 we have $m(\mathcal{O}) \neq \emptyset$. We choose an arbitrary $\varphi_0 \in m(\mathcal{O})$. A simple argument shows that $\lim_{t \to 0} d(t\varphi_0) = 0$ and hence we can choose $\delta > 0$ so that $d(\delta\varphi_0) < \varepsilon/2$.

2. According to Runge's approximation theorem there exists a sequence $\{P_k\}_{k \in \mathbb{N}}$ of polynomials with $P_k(z) \Rightarrow_{\mathcal{O}} f(z)$, hence we can choose a polynomial P with $d(P, f) < \varepsilon/2$. If we consider the function $\varphi(z) := \delta\varphi_0(z) + P(z)$ we obtain

$$d(\varphi, f) = d(\delta\varphi_0 + P, f) \leqslant d(\delta\varphi_0) + d(P, f) < \varepsilon$$

and it remains to show that $\varphi \in m(\mathcal{O})$.

To this end let be given any $\zeta \in \partial\mathcal{O}$, any compact set B with connected complement, any $g \in A(B)$ and $j \in \mathbb{Z}$. On the open set $\mathcal{O}$ denote by $\varphi^{(j)}$ the derivative of φ with order j if $j \in \mathbb{N}_0$ or an (arbitrary but fixed) antiderivative of φ with order $-j$ if $-j \in \mathbb{N}$. If $-j \in \mathbb{N}$ let $P^{(j)}$ be a fixed antiderivative with order $-j$ of the polynomial P on $\mathbb{C}$ (and hence on $\mathcal{O}$) and choose the antiderivative $\varphi_0^{(j)}$ of φ on $\mathcal{O}$ so that we have

$$\varphi^{(j)}(z) = \delta\varphi_0^{(j)}(z) + P^{(j)}(z) \qquad \text{for} \quad z \in \mathcal{O}.$$

Suppose first that $\zeta \neq \infty$. Then by the properties of φ_0 there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ with $a_n \to 0$, $b_n \to \zeta$ for $n \to \infty$ such that $a_n z + b_n \in \mathcal{O}$ for all $z \in B$, $n \in \mathbb{N}$, and

$$\varphi_0^{(j)}(a_n z + b_n) \underset{B}{\Rightarrow} \frac{1}{\delta} g(z) - \frac{1}{\delta} P^{(j)}(\zeta) \qquad (n \to \infty).$$

It follows $\varphi^{(j)}(a_n z + b_n) \Rightarrow_B g(z)$ for $n \to \infty$.

Suppose that $\zeta = \infty$ and choose a sequence $\{\zeta_m\}_{m\in\mathbb{N}}$ with $\zeta_m \in \mathbb{C}$, $\zeta_m \in \partial\mathcal{O}$, $\zeta_m \to \infty$ for $m \to \infty$. For any $m \in \mathbb{N}$ there are sequences $\{a_n^{(m)}\}_{n\in\mathbb{N}}$ and $\{b_n^{(m)}\}_{n\in\mathbb{N}}$ with $a_n^{(m)} \to 0$, $b_n^{(m)} \to \zeta_m$ for $n \to \infty$ such that $a_n^{(m)} z + b_n^{(m)} \in \mathcal{O}$ for all $z \in B$, $n \in \mathbb{N}$, and

$$\varphi_0^{(j)}(a_n^{(m)} z + b_n^{(m)}) \underset{B}{\Rightarrow} \frac{1}{\delta} g(z) - \frac{1}{\delta} \cdot P^{(j)}(\zeta_m) \qquad (n \to \infty).$$

For any $m \in \mathbb{N}$ we can choose an integer $n_m > m$ so that the following conditions hold:

$$|a_{n_m}^{(m)}| < \frac{1}{m}, \qquad |b_{n_m}^{(m)} - \zeta_m| < \frac{1}{m},$$

$$\max_B |P^{(j)}(a_{n_m}^{(m)} z + b_{n_m}^{(m)}) - P^{(j)}(\zeta_m)| < \frac{1}{m},$$

$$\max_B |\varphi_0^{(j)}(a_{n_m}^{(m)} z + b_{n_m}^{(m)}) - \frac{1}{\delta} g(z) + \frac{1}{\delta} P^{(j)}(\zeta_m)| < \frac{1}{m}.$$

It follows $\alpha_m := a_{n_m}^{(m)} \to 0$, $\beta_m := b_{n_m}^{(m)} \to \zeta = \infty$ for $m \to \infty$, and

$$\varphi^{(j)}(\alpha_m z + \beta_m) \underset{B}{\Rightarrow} g(z) \qquad \text{for} \quad m \to \infty.$$

This proves our theorem.

References

1. G. Alexits, "Konvergenzprobleme der Orthogonalreihen," VEB Deutscher Verlag der Wissenschaften, Berlin, 1960.
2. G. D. Birkhoff, Démonstratin d'une théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris* **189** (1929), 473–475.

3. C. K. Chui and M. N. Parnes, Approximation by overconvergence of a power series, *J. Math. Anal. Appl.* **36** (1971), 693–696.
4. J. B. Conway, "Functions of One Complex Variable," Springer-Verlag, New York/Heidelberg/Berlin, 1973.
5. K. Faulstich, Summierbarkeit von Potenzreihen durch Riesz-Verfahren mit komplexen Erzeugendenfolgen, *Mitt. Math. Sem. Giessen* **139** (1979).
6. K. Faulstich, W. Luh, and L. Tomm, Universelle Approximation durch Riesz-Transformierte der geometrischen Reihe, *Manuscripta Math.* **36** (1981), 309–321.
7. K. Faulstich and W. Luh, Summability of power series on prescribed sets by regular Riesz methods, *Analysis* **2** (1982), 253–265.
8. D. Gaier, "Vorlesungen über Approximation im Komplexen," Birkhäuser, Basel/Boston/Stuttgart, 1980.
9. V. I. Gavrilov, A. N. Kanatnikov and C. Pighetti, Ensembles d'accumulation généralisés, *C. R. Acad. Sci. Paris Sér. I Math.* **292** No. 7 (1981), 393–396.
10. V. I. Gavrilov and A. N. Kanatnikov, An example of a universal holomorphic function, *Dokl. Akad. Nauk SSSR* **265**, No. 2 (1982), 274–276. [Russian]
11. M. Heins, A universal Blaschke product, *Arch. Math.* **6** (1955), 41–44.
12. A. N. Kanatnikov, Limiting sets along sequences of compacta, *Dokl. Akad. Nauk SSSR* **253**, No. 1 (1980), 14–17.
13. G. G. Lorentz, Bernstein polynomials, *in* Mathematical Expositions, No. 8, University of Toronto Press, Toronto, 1953.
14. W. Luh, Approximation analytischer Funktionen durch überkonvergente Potenzreihen und deren Matrix-Transformierten, *Mitt. Math. Sem. Giessen* **88** (1970).
15. W. Luh, Über den Satz von Mergelyan, *J. Approx. Theory* **16**, No. 2 (1976), 194–198.
16. W. Luh, Über die Summierbarkeit der geometrischen Reihe, *Mitt. Math. Sem. Giessen* **113** (1974).
17. W. Luh and R. Trautner, Summierbarkeit der geometrischen Reihe auf vorgeschriebenen Mengen, *Manuscripta Math.* **18** (1976), 317–326.
18. W. Luh, Über cluster sets analytischer Funktionen, *Acta Math. Acad. Sci. Hungar.* **33**, No. 1–2 (1979), 137–142.
19. W. Luh, Universalfunktionen in einfach zusammenhängenden Gebieten, *Aequationes Math.* **19** (1979), 183–193.
20. J. Marcinkiewicz, Quelques théorèmes sur les séries orthogonales, *Ann. Soc. Polon. Math.* **16** (1938), 84–96.
21. D. Menchoff, On the partial sums of trigonometric series *Math. Sb.* **20** (62) (1974), 197–238. [Russian]
22. S. N. Mergelyan, Uniform approximations of functions of a complex variable, *Uspekhi Mat. Nauk.* (*N. S.*) **7** (2) (48) (1952), 31–122. [Russian]
23. K. Noshiro, Cluster Sets, *in* "Ergebnisse der Mathematik und ihrer Grenzgebiete," Springer-Verlag, Berlin/Göttingen/Heidelberg, 1960.
24. J. Pál, Zwei kleine Bemerkungen, *Tôhoku Math. J.* **6** (1914/15), 42–43.
25. A. Reich, Universelle Werteverteilung von Eulerprodukten, *Nachr. Akad. Wiss. Göttingen Math.-phys. Kl* II **1** (1977), 1–17.
26. A. Reich, Werteverteilung von Zetafunktionen, *Arch. Math.* **34**, No. 5 (1980), 440–451.
27. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York/Toronto/London, 1966.
28. W. J. Schneider, Approximation and harmonic measure, *in* "Aspects of Contemporary Complex Analysis, Proceeding NATO Adv. Study Inst., Univ. Durham, Durham 1979, pp. 333–349, Academic Press, London, 1980.
29. W. Seidel and J. L. Walsh, On approximation by euclidean and non-euclidean translations of analytic functions, *Bull. Amer. Math. Soc.* **47** (1941), 916–920.

30. V. I. SMIRNOV UND N. A. LEBEDEV, "Functions of a Complex Variable: Constructive Theory, MIT Press, Cambridge, 1968.
31. A. A. TALALYAN, On the a.e. convergence of the partial sums of general orthogonal series *Akad. Nauk. Armyan. SSR Dokl.* **10** (1957), 17–34. [Russian]
32. L. TOMM, "Über die Summierbarkeit der geometrischen Reihe mit regulären Verfahren, Dissertation, Ulm, 1979.
33. L. TOMM AND R. TRAUTNER, A universal power series for approximation of measurable functions, *Analysis* **2** (1–4) (1982), 1–6.
34. S. M. VORONIN, Theorem on the "Universality" of the Riemann zeta-function, *Izv. Akad. Nauk SSSR Ser. Mat.* **39** (1975), 475–486. [Russian]